Transient analysis of free-electron lasers with discrete radiators

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In the linear regime before saturation, we solve the one-dimensional, classical free-electron laser equations, maintaining the longitudinal discreteness of the electrons throughout the analysis. We then take the limit in which the beam of discrete electrons is approximated by a continuum fluid. In the continuum limit, we recover the Green function used by Wang and Yu in their treatment of self-amplified spontaneous emission (SASE). For a bunched electron beam, we discuss both incoherent and coherent SASE. We also discuss the field radiated from a bunch whose length is short compared to the radiation wavelength. [S1063-651X(99)11601-5]

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I. INTRODUCTION

Single pass free-electron lasers (FELs) have great potential as sources of high peak power radiation at short wavelengths, from the vacuum ultraviolet down to hard x rays. High gain FELs are based upon a collective instability arising from the resonant interaction between the electron beam and the radiation field. Most theoretical treatments of these devices utilize either the coupled Vlasov-Maxwell equations [1–6], or else individual particle formulations [7–11] which invoke a local average of the electron current over the electron coordinates. In both cases, the electron beam is approximated by a continuum fluid.

In the mathematical analysis of FELs, the difficulty in treating the discreteness of the electrons is due to δ function singularities in the electron current, and corresponding discontinuities in the radiated electric field at the positions of the electrons. For self-amplified spontaneous emission (SASE), the start-up from shot noise in the electron beam depends critically upon the discreteness of the electrons. Theoretical analyses of SASE [2,3,4,5,11] have incorporated the discreteness of the electrons into the initial conditions, but the electron gain medium has been approximated by a continuum fluid. In the approach developed by Wang and Yu [2], this is reflected in the fact that the Green function employed is that corresponding to a continuum electron beam. In the present paper we take a step aimed at elucidating the continuum approximation. We consider the linear regime before saturation, and solve the classical FEL equations within the one-dimensional approximation (in which the dependence of quantities on transverse coordinates is neglected), maintaining the longitudinal discreteness of the electrons throughout the analysis.

We utilize an individual particle formulation of the FEL equations introduced by Colson, Gallardo, and Bosco [8]. However, we do not invoke a local averaging approximation over longitudinal coordinates. Linearizing the one-dimensional FEL equations, we derive the third-order partial differential equation (4.6) determining the slowly varying envelope of the radiated electric field. An explicit solution is derived, corresponding to an arbitrary set of initial longitudinal coordinates of the electrons. With the solution in hand, Eqs. (4.28)–(4.30), we take the limit in which the beam of discrete electrons goes over to a continuum fluid, and we recover the results obtained in earlier work [2].

In Sec. II we review the derivation of the FEL equations. In Sec. III we discuss the one-dimensional approximation. Within this framework, energy conservation is derived and an analysis of both incoherent and coherent spontaneous radiation is presented. In Sec. IV the one-dimensional equations are linearized, and the partial differential equation (4.6), determining the slowly varying envelope E of the electric field, is derived. This equation is then solved preserving the longitudinal discreteness of the electrons. In Sec. V we take the limit in which the electron beam approaches a continuum fluid, and we discuss some useful representations of the continuum Green function. In Sec. VI we consider incoherent and coherent self-amplified spontaneous radiation for the case of a bunched electron beam, whose length is on the order of the slippage in a few gain lengths.

Because δ functions appear in the envelope equation (4.6), the envelope function E is discontinuous at the positions of the electrons. Therefore it is necessary to specify the value of E at the position of each electron. In Appendix A we use the one-dimensional FEL equations to show that the value of E at the electron position is equal to the average of the values immediately in front of and behind it. Utilizing this condition, the envelope equation becomes well defined and we can solve it.

In Sec. IV we neglect the E^* term in the envelope equation. This is the usual procedure for determining the dominant coherent growth of the radiation field. In Appendix B we extend the analysis of Sec. IV to include the effects of the E^* term. The E^* term is important when the electron bunch length is comparable to or shorter than the radiation wavelength.

In Sec. VII we discuss the field radiated from a bunch whose length is much shorter than the radiation wavelength. The analysis is based on results obtained in Appendix B. Our conclusions are given in Sec. VIII.

II. FEL EQUATIONS

We consider a highly relativistic electron beam moving in the z direction through a periodic left-handed circularly polarized helical wiggler, whose vector potential is given by

$$\vec{A}_w = A_w(\hat{e}_- e^{ik_w z} + \text{c.c.})/\sqrt{2}, \tag{2.1}$$

where $\hat{e}_{\pm}=(\hat{e}_1+i\hat{e}_2)/\sqrt{2}$, and $\hat{e}_1$ and $\hat{e}_2$ are orthogonal unit vectors transverse to $\hat{z}$. We ignore focusing in the wiggler, and assume the electron beam to have no angular spread. The transverse electron velocity is approximated by

$$\vec{v}_{\perp}\cong -e\vec{A}_w/m\gamma, \tag{2.2}$$

and the longitudinal velocity by

$$v_{\parallel}\cong c\left(1-\frac{1+K^2}{2\gamma^2}\right), \tag{2.3}$$

where $K=eA_w/mc$ is the wiggler magnetic strength parameter.

The electron beam is assumed to be initially monoenergetic with all electrons having energy γ_0 and longitudinal velocity $v_{\parallel}(\gamma_0)=v_0$. The spontaneous radiation emitted by the electrons in the forward direction is left circularly polarized with wave number k_0 and frequency $\omega_0=k_0c$,

$$\frac{k_0}{k_w}=\frac{v_0}{c-v_0}\cong\frac{2\gamma_0^2}{1+K^2} \tag{2.4}$$

and

$$k_0+k_w=\frac{\omega_0}{v_0}=\frac{\omega_w}{c-v_0}, \tag{2.5}$$

where $\omega_w=k_wc$.

We label the discrete pointlike electrons by the index $j=1,\dots,N$, where N is the total number of electrons in the beam. Choosing the axial coordinate z to be the independent variable, we denote the arrival time of the jth electron at z by $t_j(z)$, and the corresponding transverse position relative to the wiggler axis by $\vec{r}_j(z)$. Ignoring space charge effects, the radiation electric field $\vec{\varepsilon}$ is determined by the wave equation, in mks units,

$$\left(\nabla^2-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{\varepsilon}=\mu_0\frac{\partial\vec{J}_T}{\partial t}, \tag{2.6}$$

where

$$\vec{J}_T=\sum_{j=1}^{N}e(\vec{v}_{\perp})_j\delta^{(2)}[\vec{r}-\vec{r}_j(z)]\delta[t-t_j(z)]\frac{1}{c}. \tag{2.7}$$

We introduce the slowly varying envelope function $E(\vec{r},z,t)$ by

$$\vec{\varepsilon}=\frac{1}{\sqrt{2}}Ee^{ik_0z-i\omega_0t}\hat{e}_{+}+\text{c.c.} \tag{2.8}$$

The wave equation is simplified by using the paraxial approximation,

$$\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\cong 2ik_0\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right), \tag{2.9a}$$

and the resonant approximation,

$$\frac{\partial\vec{J}_T}{\partial t}\cong -i\omega_0\vec{J}_T, \tag{2.9b}$$

and we obtain

$$\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}+\frac{1}{2ik_0}\nabla_T^2\right)E =\frac{\mu_0e^2cA_w}{2m}\sum_{j=1}^{N}\frac{1}{\gamma_j(z)}e^{-i\zeta_j(z)}\delta^{(2)}(\vec{r}-\vec{r}_j(z)) \times\delta(t-t_j(z))\frac{1}{c}, \tag{2.10}$$

where we have introduced the ponderomotive phase

$$\zeta_j\equiv(k_0+k_w)z-\omega_0t_j(z) \tag{2.11}$$

of the jth electron, and ∇_T^2 is the transverse Laplacian.

The pendulum equations describing the motion of the electrons are derived as follows. Differentiating Eq. (2.11) with respect to z yields

$$\frac{d\zeta_j}{dz}=k_w\left(1-\frac{\gamma_0^2}{\gamma_j^2(z)}\right), \tag{2.12}$$

and the energy change is given by

$$\frac{d\gamma_j}{dz}=\frac{e}{mc^3}\vec{v}\cdot\vec{\varepsilon}=\frac{-e^2A_w}{2m^2c^3\gamma_j(z)}[E(z,t_j(z))e^{i\zeta_j(z)} +E^*(z,t_j(z))e^{-i\zeta_j(z)}]. \tag{2.13}$$

The coupled motion of the electrons and the radiation field is described by Eqs. (2.10)–(2.13).

III. ONE-DIMENSIONAL APPROXIMATION

In the one-dimensional approximation the sources are charged sheets of infinite transverse extent, with charge per unit area en_0'. The sheets are labeled $j=1,\dots,N$, and the energy γ_j is associated with an area $\Sigma=1/n_0'$ of the jth sheet. It is convenient to utilize the dimensionless variables [5],

$$\tau=k_wz, \tag{3.1}$$

$$\zeta=(k_0+k_w)z-\omega_0t=-k_0c\left(t-\frac{z}{v_0}\right). \tag{3.2}$$

It is worth noting that

$$\zeta-\tau=-k_0c\left(t-\frac{z}{c}\right) \tag{3.3}$$

is the phase of a wave propagating in free space. The one-dimensional FEL equations are

$$\frac{d\zeta_j(\tau)}{d\tau}=1-\frac{\gamma_0^2}{\gamma_j^2(\tau)}, \tag{3.4}$$

$$\frac{d\gamma_j(\tau)}{d\tau}=-\frac{d_2}{\gamma_j(\tau)}\,[E(\tau,\zeta_j(\tau))e^{i\zeta_j(\tau)}+E^*(\tau,\zeta_j(\tau))e^{-i\zeta_j(\tau)}], \tag{3.5}$$

$$\left(\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\zeta}\right)E(\tau,\zeta)=d_1k_0\sum_{j=1}^{N}\frac{e^{-i\zeta_j(\tau)}}{\gamma_j(\tau)}\,\delta(\zeta-\zeta_j(\tau)), \tag{3.6}$$

with

$$d_1=\frac{n_0'\mu_0e^2cA_w}{2mk_w}, \tag{3.7}$$

$$d_2=\frac{e^2A_w}{2m^2c^3k_w}. \tag{3.8}$$

In deriving Eq. (3.6) from Eq. (2.10), a local average has been carried out over the transverse coordinates. Also, the transverse Laplacian has been dropped, corresponding to ignoring diffraction.

Energy conservation is derived by multiplying Eq. (3.6) by $E^*(\tau,\zeta)$ and adding the result to its complex conjugate:

$$\left(\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\zeta}\right)|E(\tau,\zeta)|^2=d_1k_0\sum_{j=1}^{N}\frac{1}{\gamma_j(\tau)}\,[E(\tau,\zeta_j(\tau))e^{i\zeta_j(\tau)}+E^*(\tau,\zeta_j(\tau))e^{-i\zeta_j(\tau)}]\,\delta(\zeta-\zeta_j(\tau)). \tag{3.9}$$

Integrating Eq. (3.9) over ζ, and using Eq. (3.5), we derive

$$\sum_{j=1}^{N}\frac{d\gamma_j(\tau)}{d\tau}=-\frac{d_2}{d_1k_0}\frac{d}{d\tau}\int|E(\tau,\zeta)|^2d\zeta. \tag{3.10}$$

Since

$$\frac{d_2}{d_1}=\frac{\varepsilon_0}{n_0'mc^2}, \tag{3.11}$$

Eq. (3.10) implies

$$n_0'mc^2\sum_{j=1}^{N}\gamma_j(\tau)+\int\varepsilon_0|E(\tau,\zeta)|^2\frac{d\zeta}{k_0}=\text{const}, \tag{3.12}$$

which is the expression of energy conservation.

Insight into the one-dimensional approximation is obtained by considering the spontaneous radiation. In this case the paraxial wave equation (3.6) is simplified by setting $\zeta_j(\tau)=\zeta_j(0)$ and $\gamma_j(\tau)=\gamma_j(0)=\gamma_0$ on the right-hand side. The solution of the resulting equation is

$$E(\tau,\zeta)=\frac{d_1k_0}{\gamma_0}\sum_{j=1}^{N}e^{-i\zeta_j(0)}S(\tau,\zeta-\zeta_j(0)), \tag{3.13}$$

where

$$S(\tau,\zeta-\zeta_j(0))=\begin{cases}1, & 0<\zeta-\zeta_j(0)<\tau\\ \frac{1}{2}, & \zeta=\zeta_j(0)\\ 0 & \text{otherwise.}\end{cases} \tag{3.14}$$

The electric field is discontinuous at the position of the source, $\zeta=\zeta_j(0)$, and its value at the source position is the average of the values just in front of and behind it (see Appendix A).

From Eq. (3.10), the radiated energy loss $\Delta\Gamma(\tau)$ from a bunch of N electrons is given by

$$\Delta\Gamma(\tau)\equiv\sum_{j=1}^{N}[\gamma_j(\tau)-\gamma_0]=-\frac{d_2}{d_1k_0}\int d\zeta|E(\tau,\zeta)|^2. \tag{3.15}$$

Multiplying Eq. (3.13) by its complex conjugate, we find

$$|E(\tau,\zeta)|^2=\frac{d_1^2k_0^2}{\gamma_0^2}\left[\sum_{j=1}^{N}|S(\tau,\zeta-\zeta_j(0))|^2+\sum_{j\neq l}e^{-i\zeta_j(0)}e^{i\zeta_l(0)}S(\tau,\zeta-\zeta_j(0))\times S^*(\tau,\zeta-\zeta_l(0))\right]. \tag{3.16}$$

For the moment, let us suppose that the number of sources in the interval $(\zeta_0,\zeta_0+d\zeta_0)$ is well approximated by

$$\frac{n_1}{k_0}D(\zeta_0)d\zeta_0, \tag{3.17a}$$

where $D(\zeta_0)$ is a smooth function with $0\leqslant D(\zeta_0)\leqslant 1$. The quantity n_1 is the maximum line density, locally averaged to eliminate the high frequency shot noise.

$$N=\int\frac{n_1}{k_0}D(\zeta_0)d\zeta_0. \tag{3.17b}$$

Using the distribution $D(\zeta_0)$, the sums in Eq. (3.16) can be approximated by integrals, and from Eq. (3.15) it follows that

$$\langle\Delta\Gamma(\tau)\rangle=-\frac{d_1d_2k_0}{\gamma_0^2}\left[\frac{n_1}{k_0}\int d\zeta\int d\zeta_0D(\zeta_0)|S(\tau,\zeta-\zeta_0)|^2+\left(\frac{n_1}{k_0}\right)^2\int d\zeta\left|\int d\zeta_0D(\zeta_0)e^{-i\zeta_0}S(\tau,\zeta-\zeta_0)\right|^2\right]. \tag{3.18}$$

Utilizing the Fourier expansion,

$$S(\tau,\zeta-\zeta_0)=\int\frac{dq}{2\pi}\,e^{iq(\zeta-\zeta_0)}\int_0^\tau d\zeta'\,e^{-iq\zeta'}, \tag{3.19}$$

and introducing the Fourier transform

$$\tilde{D}(k)=\int d\zeta_0D(\zeta_0)e^{-ik\zeta_0}, \tag{3.20}$$

we reexpress Eq. (3.18) in the form

$$\langle \Delta\Gamma(\tau)\rangle = \frac{-d_1 d_2 k_0}{\gamma_0^2}\left[N\int \frac{dq}{2\pi}\frac{4\sin^2(q\tau/2)}{q^2} + N^2\int \frac{dq}{2\pi}\left|\frac{\tilde{D}(1+q)}{\tilde{D}(0)}\right|^2 \frac{4\sin^2(q\tau/2)}{q^2}\right]. \tag{3.21}$$

The first term on the right-hand side is the incoherent spontaneous radiation, and the second term is the coherent spontaneous radiation. The integral over q in the first term can be evaluated, and the incoherent energy loss is found to be

$$\Delta\Gamma_{\rm inc}(\tau) = -\frac{N d_1 d_2 k_0 \tau}{\gamma_0^2}. \tag{3.22}$$

For point electrons, it is well known that in the forward direction for $\omega \approx \omega_0$ the spontaneous radiated energy per unit solid angle, per unit frequency interval is given by [11]

$$\frac{dI(\omega)}{d\Omega} = \frac{1}{4\pi\varepsilon_0}\frac{e^2 k_0^2 K^2}{2\pi^2 c k_w^2 \gamma_0^2}\frac{\sin^2[\pi N_w(\omega/\omega_0 - 1)]}{(\omega/\omega_0 - 1)^2}. \tag{3.23}$$

Using $K = eA_w/mc$, and the definitions of d_1 and d_2 given in Eqs. (3.7) and (3.8), we find

$$\frac{N}{mc^2}\frac{dI(\omega)}{d\Omega}\frac{\lambda_0^2}{\Sigma}d\omega = \frac{N d_1 d_2 k_0}{\gamma_0^2}\frac{4\sin^2(q\tau/2)}{2\pi q^2}dq, \tag{3.24}$$

where $\lambda_0 = 2\pi/k_0$ is the resonant radiation wavelength, $\Sigma = 1/n_0'$, and $q = (\omega - \omega_0)/\omega_0$ is the frequency detuning. From Eqs. (3.24) and (3.21), it follows that the radiated energy loss $\Delta\gamma(\tau)$, computed within the one-dimensional approximation, corresponds to the spontaneous radiation emitted by point electrons into the solid angle [11,12]

$$\Delta\Omega = \lambda_0^2/\Sigma. \tag{3.25}$$

To conclude this section, let us review the conditions [5,10,12] for the validity of the one-dimensional approximation. Consider a long wiggler in which there is exponential gain of the radiation field. We denote the gain length (e-folding length) of the electric field by L_G, and suppose the electron beam to have a circular cross section of area Σ_e. Diffraction will have a negligible effect on the gain if the Rayleigh range corresponding to the electron beam cross section is greater than or on the order of the gain length, i.e.,

$$\frac{\Sigma_e}{\lambda_0} \gtrsim L_G. \tag{3.26}$$

On the other hand, when the Rayleigh range is short compared to the gain length, diffraction will reduce the gain below that given by the one-dimensional approximation.

When the Rayleigh range is long compared to the gain length, there is not communication between all parts of the electron beam cross section, and consequently, the output radiation will be comprised of many transverse modes [5]. For a parallel electron beam, the one-dimensional approximation provides a good description when the gain is not reduced by diffraction and there is only one transverse mode, i.e., when the Rayleigh range is approximately equal to the gain length [5],

$$\frac{\Sigma_e}{\lambda_0} \approx L_G. \tag{3.27}$$

It is worth noting that the condition, Eq. (3.27), for the validity of the one-dimensional approximation can be rewritten in the form

$$\frac{\lambda_0^2}{\Sigma_e} \approx \frac{\lambda_0}{L_G}. \tag{3.28}$$

Equation (3.28) has the physical interpretation that the diffraction angle corresponding to the electron beam transverse dimension is approximately equal to the angle $\sqrt{\lambda_0/L_G}$ characteristic of radiation from a section of undulator of extent equal to one gain length.

IV. LINEAR REGIME

There is an important regime before saturation in which the FEL equations can be linearized. We write [8]

$$\zeta_j(\tau) = \zeta_j(0) + \dot{\zeta}_j(0)\tau + \delta\zeta_j(\tau), \tag{4.1}$$

where the dot indicates differentiation with respect to τ. We consider an initially monoenergetic electron beam, so $\gamma_j(0) = \gamma_0$ for all j. Hence, it follows from Eq. (3.4) that $\dot{\zeta}_j(0) = 0$ for all j. When $|\gamma_j(\tau) - \gamma_0| \ll \gamma_0$, Eq. (3.4) becomes

$$\dot{\zeta}_j(\tau) \cong 2\left(\frac{\gamma_j(\tau) - \gamma_0}{\gamma_0}\right) \tag{4.2}$$

and

$$\ddot{\zeta}_j(\tau) = \delta\ddot{\zeta}_j(\tau) \cong \frac{2}{\gamma_0}\dot{\gamma}_j(\tau). \tag{4.3}$$

Using Eq. (3.5) in Eq. (4.3) yields [8]

$$\delta\zeta_j(\tau) \cong -\frac{2d_2}{\gamma_0^2}\int_0^\tau d\tau''(\tau - \tau'')[E(\tau'', \zeta_j(0))e^{i\zeta_j(0)} + E^*(\tau'', \zeta_j(0))e^{-i\zeta_j(0)}], \tag{4.4}$$

where we have kept only terms linear in E.

We approximate the paraxial wave equation (3.6), by the linear equation

$$\left(\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\zeta}\right)E(\tau,\zeta) = \frac{d_1 k_0}{\gamma_0}\sum_{j=1}^{N} e^{-i\zeta_j(0)}[1 - i\,\delta\zeta_j(\tau)] \times \delta(\zeta - \zeta_j(0)). \tag{4.5}$$

Differentiating this equation twice with respect to τ (holding ζ fixed) and using Eq. (4.4) for $\delta\zeta_j(\tau)$ leads to

$$\frac{\partial^2}{\partial \tau^2}\left(\frac{\partial}{\partial \tau}+\frac{\partial}{\partial \zeta}\right)E(\tau,\zeta) = i\alpha \frac{k_0}{n_1}\sum_{j=1}^{N}[E(\tau,\zeta_j(0))+E^*(\tau,\zeta_j(0))e^{-2i\zeta_j(0)}] \times \delta(\zeta-\zeta_j(0)). \tag{4.6}$$

In Eq. (4.6), the quantity α is defined by

$$\alpha=(2\rho)^3=\frac{2n_1d_1d_2}{\gamma_0^3}, \tag{4.7}$$

where ρ is the Pierce parameter [9], and n_1 is the maximum of the line density (number of charged sheets per unit length), computed by carrying out a local average to eliminate the high frequency shot noise. Note that α depends only on the number of electrons per unit volume n_0,

$$n_0=n_1n_0'. \tag{4.8}$$

We wish to determine the coherent growth of the slowly varying amplitude E, defined in Eq. (2.8). For this purpose, we can neglect the second term on the right-hand side of Eq. (4.6), which is proportional to E^*. (The E^* term is discussed in Appendix B.) Therefore we shall consider the equation

$$\frac{\partial^2}{\partial \tau^2}\left(\frac{\partial}{\partial \tau}+\frac{\partial}{\partial \zeta}\right)E(\tau,\zeta)=i\alpha\frac{k_0}{n_1}\sum_{j=1}^{N}E(\tau,\zeta_j(0))\delta(\zeta-\zeta_j(0)). \tag{4.9}$$

The electric field is discontinuous at the positions of the charged sheets, $\zeta=\zeta_j(\tau)\cong\zeta_j(0)$, so Eq. (4.9) must be supplemented by (see Appendix A)

$$E(\tau,\zeta_j(0))=\tfrac{1}{2}[E(\tau,\zeta_j(0)+)+E(\tau,\zeta_j(0)-)]. \tag{4.10}$$

To solve Eq. (4.6), we apply the Laplace transform

$$F(s,\zeta)=\int_0^\infty d\tau\, e^{-s\tau}E(\tau,\zeta), \tag{4.11}$$

and derive

$$\frac{\partial}{\partial \zeta}F(s,\zeta)+sF(s,\zeta) -\frac{2\sigma^2}{s^2}\sum_{j=1}^{N}F(s,\zeta_j(0))\delta(\zeta-\zeta_j(0))=H(s,\zeta), \tag{4.12}$$

where

$$\sigma^2=\frac{i\alpha}{2}\frac{k_0}{n_1} \tag{4.13}$$

and

$$H(s,\zeta)=E(0,\zeta)+\frac{1}{s}\left[\left(\frac{\partial}{\partial \tau}+\frac{\partial}{\partial \zeta}\right)E(\tau,\zeta)\right]_{\tau=0}. \tag{4.14}$$

Defining $E_0(\zeta)=E(0,\zeta)$, and using Eq. (3.6), we rewrite Eq. (4.14) in the form

$$H(s,\zeta)=E_0(\zeta)+\frac{d_1k_0}{\gamma_0 s}\sum_{j=1}^{N}e^{-i\zeta_j(0)}\delta(\zeta-\zeta_j(0)). \tag{4.15}$$

We write

$$F(s,\zeta)=e^{-s\zeta}f(s,\zeta), \tag{4.16}$$

and insert this expression in Eq. (4.12) to obtain

$$\frac{\partial}{\partial \zeta}f(s,\zeta)=\frac{2\sigma^2}{s^2}\sum_{j=1}^{N}f(s,\zeta_j(0))\delta(\zeta-\zeta_j(0))+e^{s\zeta}H(s,\zeta). \tag{4.17}$$

We specify the initial phases of the charged sheets,

$$-\infty<\zeta_1(0)<\zeta_2(0)<\cdots<\zeta_N(0)<\infty, \tag{4.18}$$

and adopt the convention $\zeta_0(0)=-\infty$ and $\zeta_{N+1}(0)=\infty$. We require $f(s,\zeta)\to 0$ as $\zeta\to-\infty$.

Equation (4.17) implies

$$f(s,\zeta_j(0)+)-f(s,\zeta_j(0)-) =\frac{2\sigma^2}{s^2}f(s,\zeta_j(0))+\frac{d_1k_0}{\gamma_0 s}e^{s\zeta_j(0)}e^{-i\zeta_j(0)}, \tag{4.19}$$

and Eq. (4.10) implies

$$f(s,\zeta_j(0))=\tfrac{1}{2}[f(s,\zeta_j(0)+)+f(s,\zeta_j(0)-)]. \tag{4.20}$$

Using Eq. (4.20) on the right-hand side of Eq. (4.19) yields

$$f(s,\zeta_j(0)+)=\left(\frac{1+\sigma^2/s^2}{1-\sigma^2/s^2}\right)f(s,\zeta_j(0)-) +\frac{d_1k_0}{\gamma_0 s}(1-\sigma^2/s^2)^{-1}e^{s\zeta_j(0)}e^{-i\zeta_j(0)}. \tag{4.21}$$

From Eq. (4.17), we observe

$$f(s,\zeta_j(0)-)=f(s,\zeta_{j-1}(0)+)+\int_{\zeta_{j-1}(0)}^{\zeta_j(0)}d\zeta'\,e^{s\zeta'}E_0(\zeta'). \tag{4.22}$$

With the initial condition

$$f(s,\zeta_0(0)+)=0 \tag{4.23}$$

the recursion relations given in Eqs. (4.21) and (4.22) determine $f(s,\zeta_j(0)+)$ for all j. Next, $f(s,\zeta)$ can be found for arbitrary ζ by noting that for $\zeta_j(0)<\zeta<\zeta_{j+1}(0)$:

$$f(s,\zeta)=f(s,\zeta_j(0)+)+\int_{\zeta_j(0)}^{\zeta}d\zeta'\,e^{s\zeta'}E_0(\zeta'). \tag{4.24}$$

From Eqs. (4.21)–(4.24), it follows that, for $\zeta_j(0)<\zeta<\zeta_{j+1}(0)$,

$$f(s,\zeta)=\sum_{l=1}^{j}\left(\frac{1+\sigma^2/s^2}{1-\sigma^2/s^2}\right)^{j-l+1}\int_{\zeta_{l-1}(0)}^{\zeta_l(0)}d\zeta' e^{s\zeta'}E_0(\zeta')+\int_{\zeta_j(0)}^{\zeta}d\zeta' e^{s\zeta'}E_0(\zeta')+\frac{d_1k_0}{\gamma_0 s}\times\sum_{l=1}^{j}\left(\frac{1+\sigma^2/s^2}{1-\sigma^2/s^2}\right)^{j-l}(1-\sigma^2/s^2)^{-1}e^{s\zeta_j(0)}e^{-i\zeta_j(0)}. \tag{4.25}$$

Using Eqs. (4.25) and (4.16), the Laplace transform $F(s,\zeta)$ of the slowly varying envelope $E(\tau,\zeta)$ is found to be given by

$$F(s,\zeta)=\int_{-\infty}^{\infty}d\zeta'\,\theta(\zeta-\zeta')e^{s(\zeta'-\zeta)}\left(\frac{1+\sigma^2/s^2}{1-\sigma^2/s^2}\right)^{m(\zeta,\zeta')}\times E_0(\zeta')+\frac{d_1k_0}{\gamma_0 s}\sum_{j=1}^{N}e^{-\zeta_j(0)}\theta(\zeta-\zeta_j(0))\times e^{s(\zeta_j(0)-\zeta)}(1-\sigma^2/s^2)^{-1}\left(\frac{1+\sigma^2/s^2}{1-\sigma^2/s^2}\right)^{m(\zeta,\zeta_j(0)+)}. \tag{4.26}$$

The step function $\theta(\zeta-\zeta')$ is unity for $\zeta>\zeta'$ and vanishes for $\zeta<\zeta'$. The integer valued function $m(\zeta,\zeta')$ is equal to the number of charged sheets between ζ' and ζ.

The envelope $E(\tau,\zeta)$ is determined by employing the inverse Laplace transform,

$$E(\tau,\zeta)=\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}\,e^{s\tau}F(s,\zeta), \tag{4.27}$$

where the contour of integration is to the right of all singularities of $F(s,\zeta)$. Inserting Eq. (4.26) into Eq. (4.27) yields the result

$$E(\tau,\zeta)=\int_{-\infty}^{\infty}d\zeta' g_2(\tau,\zeta,\zeta')E_0(\zeta')+\frac{d_1k_0}{\gamma_0}\sum_{j=1}^{N}g_1(\tau,\zeta,\zeta_j(0))e^{-i\zeta_j(0)}, \tag{4.28}$$

with

$$g_2(\tau,\zeta,\zeta')=\theta(\zeta-\zeta')\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}\,e^{s(\tau-\zeta+\zeta')}\times\left(\frac{1+\sigma^2/s^2}{1-\sigma^2/s^2}\right)^{m(\zeta,\zeta')} \tag{4.29}$$

and

$$g_1(\tau,\zeta,\zeta')=\theta(\zeta-\zeta')\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi is}\,e^{s(\tau-\zeta+\zeta')}\times(1-\sigma^2/s^2)^{-1}\left(\frac{1+\sigma^2/s^2}{1-\sigma^2/s^2}\right)^{m(\zeta,\zeta'+)}. \tag{4.30}$$

The integrals in Eqs. (4.29) and (4.30) vanish when $\tau-\zeta+\zeta'<0$, since in this case the contour can be closed in the right half plane, which contains no singularities. It follows that $g_{1,2}(\tau,\zeta,\zeta')$ are nonvanishing only in the interval $\zeta-\tau<\zeta'<\zeta$.

In Appendix C we briefly discuss the evaluation of $g_1(\tau,\zeta,\zeta')$ as given in Eq. (4.30).

V. APPROXIMATING ELECTRON BEAM BY CONTINUUM FLUID

Let us now consider approximating the electron gain medium by a continuum fluid. We suppose that the number of charged sheets per unit length n_1 increases toward infinity, while the charge per unit area en_0' on each sheet decreases toward zero, with the number of electrons per unit volume n_0 held fixed:

$$n_1\to\infty, n_0'\to 0 \quad \text{with } n_1n_0'=n_0 \text{ fixed.} \tag{5.1}$$

In this limit we assume that the number of charged sheets in the interval $(\zeta_0,\zeta_0+d\zeta_0)$ is given by

$$\frac{n_1}{k_0}D(\zeta_0)d\zeta_0, \tag{5.2}$$

where $D(\zeta_0)$ is a smooth function with $0\leqslant D(\zeta_0)\leqslant 1$.

The number of charged sheets between ζ' and ζ has the limiting behavior

$$m(\zeta,\zeta')\to\frac{n_1}{k_0}\int_{\zeta'}^{\zeta}d\zeta_0 D(\zeta_0). \tag{5.3}$$

The quantity $\alpha=(2\rho)^3$ defined in Eq. (4.7) depends only on n_0, and hence remains constant, while σ^2 defined in Eq. (4.13) vanishes according to

$$\sigma^2=\frac{i\alpha}{2}\frac{k_0}{n_1}. \tag{5.4}$$

It is now clear that

$$\left(\frac{1+\sigma^2/s^2}{1-\sigma^2/s^2}\right)^{m(\zeta,\zeta')}\to\left(\frac{1+(i\alpha/2s^2)(k_0/n_1)}{1-(i\alpha/2s^2)(k_0/n_1)}\right)^{(n_1/k_0)\int_{\zeta'}^{\zeta}d\zeta_0 D(\zeta_0)}\to\exp\left[\frac{i\alpha}{s^2}\int_{\zeta'}^{\zeta}d\zeta_0 D(\zeta_0)\right]. \tag{5.5}$$

The functions g_1 and g_2 [Eqs. (4.29) and (4.30)] have the limits

$$g_1(\tau,\zeta,\zeta')\to\dot{g}(\tau,\zeta,\zeta') \tag{5.6}$$

and

$$g_2(\tau,\zeta,\zeta')\to\ddot{g}(\tau,\zeta,\zeta'), \tag{5.7}$$

where the dots denote derivatives with respect to τ of the continuum Green function g [2], given by

$$g(\tau,\zeta,\zeta')=\theta(\zeta-\zeta')\times\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i s^2}\,e^{s(\tau-\zeta+\zeta')}e^{i\alpha/s^2\int_{\zeta'}^{\zeta}d\zeta_0 D(\zeta_0)}. \tag{5.8}$$

The continuum Green function satisfies the equation [2,5]

$$\left[\frac{\partial^2}{\partial\tau^2}\left(\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\zeta}\right)-i\alpha D(\zeta)\right]g(\tau,\zeta,\zeta')=\delta(\tau)\delta(\zeta-\zeta'). \tag{5.9}$$

Taylor expanding the exponential functions in Eq. (5.8), and summing the contributions of the poles at $s=0$ [10,13], one finds

$$\dot{g}(\tau,\zeta,\zeta')=S(\tau,\zeta-\zeta')\times\sum_{l=0}^{\infty}[i\alpha w(\zeta,\zeta')]^l\,\frac{(\tau-\zeta+\zeta')^{2l}}{(2l)!}\,\frac{(\zeta-\zeta')^l}{l!} \tag{5.10}$$

and

$$\ddot{g}(\tau,\zeta,\zeta')=\delta(\tau-\zeta+\zeta')+S(\tau,\zeta-\zeta')\times\sum_{l=1}^{\infty}[i\alpha w(\zeta,\zeta')]^l\,\frac{(\tau-\zeta+\zeta')^{2l-1}}{(2l-1)!}\,\frac{(\zeta-\zeta')^l}{l!}, \tag{5.11}$$

where

$$w(\zeta,\zeta')=\frac{1}{\zeta-\zeta'}\int_{\zeta'}^{\zeta}d\zeta_0 D(\zeta_0). \tag{5.12}$$

The function $S(\tau,\zeta-\zeta')$ was defined in Eq. (3.14) during our earlier discussion of spontaneous radiation.

The collective behavior in the FEL becomes apparent when we reexpress Eq. (5.8) in the equivalent form

$$g(\tau,\zeta,\zeta')=\int_{c-i\infty}^{c+i\infty}\frac{ds e^{s\tau}}{2\pi i}\int_{-\infty-iQ}^{\infty-iQ}\frac{dq}{2\pi}\,\frac{e^{iq(\zeta-\zeta')}}{s^3+iqs^2-i\alpha w(\zeta,\zeta')}, \tag{5.13}$$

where the contour of the q integration is taken to lie below the pole, assuring the integral vanishes when $\zeta-\zeta'<0$.

We denote by s_1,s_2,s_3 the three solutions of the cubic equation

$$s^3+iqs^2-i\alpha w(\zeta,\zeta')=0. \tag{5.14}$$

Interchanging the order of the integrations in Eq. (5.13), we evaluate the s integration in terms of the residues of the three poles, obtaining

$$g(\tau,\zeta,\zeta')=\int_{-\infty-iQ}^{\infty-iQ}\frac{dq}{2\pi}\,e^{iq(\zeta-\zeta')}G(\tau,q;w(\zeta,\zeta')), \tag{5.15}$$

with

$$G(\tau,q;w(\zeta,\zeta'))=\frac{e^{s_1\tau}}{(s_1-s_2)(s_1-s_3)}+\frac{e^{s_2\tau}}{(s_2-s_1)(s_2-s_3)}+\frac{e^{s_3\tau}}{(s_3-s_1)(s_3-s_2)}. \tag{5.16}$$

Recall that s_1,s_2,s_3 are functions of q and $w(\zeta,\zeta')$, as determined by the cubic equation (5.14).

Equations (5.14)–(5.16) provide a generalization to the bunched electron beam of the well known result for an unbunched (coasting) electron beam, for which $w(\zeta,\zeta')=1$. For the coasting beam, it is a consequence of translation invariance that the Green function has the form $g(\tau,\zeta-\zeta')$, and s_1,s_2,s_3 are independent of ζ and ζ'. In this case Eq. (5.15) is a Fourier representation in the variable $\zeta-\zeta'$. One generally lets μ denote one of the cube roots of i,

$$\mu=e^{i\pi/6},\quad e^{i5\pi/6},\quad e^{i3\pi/2}, \tag{5.17}$$

and expresses the solution of the cubic equation (5.14) in the perturbation expansion:

$$\frac{s}{2\rho}=\mu-\frac{i}{3}\left(\frac{q}{2\rho}\right)-\frac{1}{9\mu}\left(\frac{q}{2\rho}\right)^2+\cdots. \tag{5.18}$$

The fastest growing mode [$\exp(s_1\tau)$] corresponds to

$$\frac{s_1}{2\rho}\approx\frac{\sqrt{3}}{2}\left[1-\frac{1}{9}\left(\frac{q}{2\rho}\right)^2\right]+i\left[\frac{1}{2}-\frac{1}{3}\left(\frac{q}{2\rho}\right)+\frac{1}{18}\left(\frac{q}{2\rho}\right)^2\right]. \tag{5.19}$$

Use of Eq. (5.19) in Eq. (5.16) yields the widely used Gaussian approximation to the dependence of $G(\tau,q)$ on the detuning $q=(\omega-\omega_0)/\omega_0$. This approach is not so helpful in the case of a bunched electron beam, and we base our analysis in the next section on the power series representation given in Eq. (5.10).

VI. RADIATED ENERGY

For amplified spontaneous emission, we showed in Eq. (4.28) that the electric field envelope can be expressed as [2,5]

$$E(\tau,\zeta)=\frac{d_1k_0}{\gamma_0}\sum_{j=1}^{N}e^{-i\zeta_j(0)}g_1(\tau,\zeta,\zeta_j(0)). \tag{6.1}$$

The corresponding radiated energy loss $\Delta\Gamma(\tau)$ from a bunch of N electrons was found in Eq. (3.15) to be determined by

$$\Delta\Gamma(\tau)=-\frac{d_2}{d_1k_0}\int d\zeta|E(\tau,\zeta)|^2. \tag{6.2}$$

The square of the magnitude of the envelope function is found by multiplying Eq. (6.1) by its complex conjugate:

$$|E(\tau,\zeta)|^2=\frac{d_1^2k_0^2}{\gamma_0^2}\Bigg[\sum_{j=1}^{N}|g_1(\tau,\zeta,\zeta_j(0))|^2+\sum_{j\neq l}e^{-i\zeta_j(0)}e^{i\zeta_l(0)}g_1(\tau,\zeta,\zeta_j(0))\times g_1^*(\tau,\zeta,\zeta_l(0))\Bigg]. \tag{6.3}$$

Given the initial phases $\{\zeta_j(0)\}$ of the electrons, Eqs. (6.2) and (6.3) can be used to calculate the radiated energy. If one thinks of $\{\zeta_j(0)\}$ as stochastic variables, the average radiated energy, its fluctuation, and other statistical properties can be determined [14–16]. Here, we shall not analyze the fluctuation and statistical properties of the radiation, but shall consider the average energy radiated by a bunch of electrons.

We suppose the average properties of the electron beam are adequately described by the distribution $D(\zeta_0)$ introduced in Eq. (5.2). Also, we approximate $g_1(\tau,\zeta,\zeta')$ by its continuum limit $\dot{g}(\tau,\zeta,\zeta')$, as discussed in Eq. (5.6). The function $\dot{g}(\tau,\zeta,\zeta')$ will be evaluated using the power series expansion of Eq. (5.10). The average radiated energy is given by

$$\langle\Delta\Gamma(\tau)\rangle=-\frac{d_1d_2k_0}{\gamma_0^2}\Bigg[\frac{n_1}{k_0}\int d\zeta\int d\zeta_0D(\zeta_0)|\dot{g}(\tau,\zeta,\zeta_0)|^2+\left(\frac{n_1}{k_0}\right)^2\int d\zeta\left|\int d\zeta_0D(\zeta_0)e^{-i\zeta_0}\dot{g}(\tau,\zeta,\zeta_0)\right|^2\Bigg]. \tag{6.4}$$

For a short wiggler, this expression reduces to Eq. (3.18) derived during our discussion of spontaneous radiation. In Eq. (6.4) the first term corresponds to incoherent SASE and the second term to coherent SASE.

At this point it is useful to introduce the scaled variables employed by Bonifacio, Pellegrini, and Narducci [9],

$$\bar{\tau}=2\rho\tau,\quad \bar{\zeta}=2\rho\zeta,\quad \text{and}\quad \bar{\zeta}'=2\rho\zeta'. \tag{6.5}$$

Here, ρ is the Pierce parameter defined in Eq. (4.7). Let us assume the electron bunch distribution has the form

$$D(\zeta_0;\zeta_B)=\Delta(\zeta_0/\zeta_B)=D(\bar{\zeta}_0;\bar{\zeta}_B), \tag{6.6}$$

where ζ_B parametrizes the bunch length, $\bar{\zeta}_0=2\rho\zeta_0$ and $\bar{\zeta}_B=2\rho\zeta_B$. From Eq. (5.12) it follows that

$$w(\zeta,\zeta';\zeta_B)=\frac{1}{\zeta-\zeta'}\int_{\zeta'}^{\zeta}d\zeta_0D(\zeta_0;\zeta_B)=w(\bar{\zeta},\bar{\zeta}';\bar{\zeta}_B). \tag{6.7}$$

Using $\alpha=(2\rho)^3$ [Eq. (4.7)] and Eq. (6.7) in Eq. (5.10), we can express the response function in terms of the scaled variables,

$$\dot{g}(\tau,\zeta,\zeta')=h(\bar{\tau},\bar{\zeta},\bar{\zeta}';\bar{\zeta}_B)=S(\bar{\tau},\bar{\zeta}-\bar{\zeta}')\sum_{l=0}^{\infty}[iw(\bar{\zeta},\bar{\zeta}';\bar{\zeta}_B)]^l\times\frac{(\bar{\tau}-\bar{\zeta}+\bar{\zeta}')^{2l}}{(2l)!}\frac{(\bar{\zeta}-\bar{\zeta}')^l}{l!}. \tag{6.8}$$

The expression [Eq. (6.4)] for the average radiated energy can also be rewritten using the scaled variables. We find

$$\langle\Delta\Gamma\rangle=\frac{-d_1d_2k_0}{2\rho\gamma_0^2}\Bigg[\frac{n_1}{2\rho k_0}\int d\bar{\zeta}\int d\bar{\zeta}_0D(\bar{\zeta}_0;\bar{\zeta}_B)\times|h(\bar{\tau},\bar{\zeta},\bar{\zeta}_0;\bar{\zeta}_B)|^2+\left(\frac{n_1}{2\rho k_0}\right)^2\times\int d\bar{\zeta}\left|\int d\bar{\zeta}_0D(\bar{\zeta}_0;\bar{\zeta}_B)e^{-i\bar{\zeta}_0/2\rho}h(\bar{\tau},\bar{\zeta},\bar{\zeta}_0;\bar{\zeta}_B)\right|^2\Bigg]. \tag{6.9}$$

Let us express Eq. (6.9) in the following form:

$$\langle\Delta\Gamma\rangle=(\Delta\Gamma_g)\Bigg[N_{eg}\int d\bar{\zeta}\Gamma_{\rm inc}(\bar{\tau},\bar{\zeta};\bar{\zeta}_B)+N_{eg}^2\int d\bar{\zeta}\Gamma_{\rm coh}(\bar{\tau},\bar{\zeta};\bar{\zeta}_B,\rho)\Bigg]. \tag{6.10}$$

Here,

$$\Delta\Gamma_g=-\frac{d_1d_2k_0}{2\rho\gamma_0^2} \tag{6.11}$$

is the energy lost by one electron in the first gain length of the wiggler, and

$$N_{eg}=\frac{n_1}{2\rho k_0} \tag{6.12}$$

is the (peak) number of electrons a radiation wave front slips over in the time it takes an electron to travel one gain length down the wiggler.

In Figs. 1 and 2, respectively, we plot the quantities $\Gamma_{\rm inc}(\bar{\tau},\bar{\zeta};\bar{\zeta}_B)$ and $\Gamma_{\rm coh}(\bar{\tau},\bar{\zeta};\bar{\zeta}_B,\rho)$. In these figures the scaled distance along the wiggler axis at which the energy loss is computed is taken to be $\bar{\tau}=5$. The bunch profile is chosen to be the step function distribution:

$$D(\bar{\zeta}_0;\bar{\zeta}_B)=\begin{cases}1, & 0<\bar{\zeta}_0<\bar{\zeta}_B\\ 0, & \text{otherwise,}\end{cases} \tag{6.13}$$

with the scaled bunch length $\bar{\zeta}_B=2$. The Pierce parameter is $\rho=\frac{1}{40}$.

When considering Figs. 1 and 2, one should recall that radiation arrives at $\bar{\tau}=5$ during a time interval characterized by $0<\bar{\zeta}<7$. The back of the electron bunch is located at $\bar{\zeta}=0$ and the front at $\bar{\zeta}=2$. Radiation in the interval $2<\bar{\zeta}<7$ has slipped out of the electron bunch, while that in $0<\bar{\zeta}<2$ is located within the bunch. Radiation near $\bar{\zeta}=0$ has

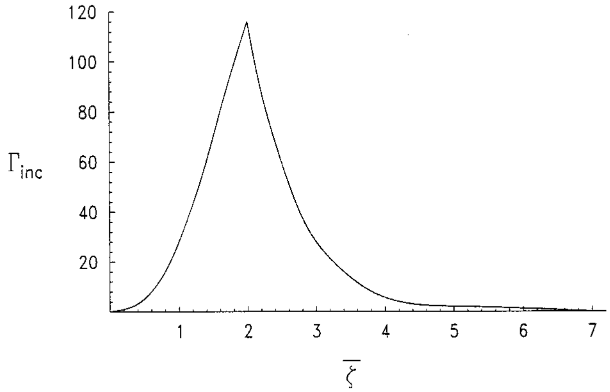

FIG. 1. We plot $\Gamma_{inc}(\overline{\tau},\overline{\zeta};\overline{\zeta_B})$ against $\overline{\zeta}$, for $\overline{\tau}=5$ and $\overline{\zeta_B}=2$. The integral under the curve is 163. The scaled quantities plotted are dimensionless.

not been amplified, since it has just been emitted and has not had time to slip through the bunch. Near $\overline{\zeta}=7$ the radiation emitted by electrons at the front of the bunch immediately after they entered the wiggler is found. This radiation is unamplified, because it was emitted before the FEL action had produced a density modulation on the electron beam.

In Fig. 1 the integral over Γ_{inc} with respect to $\overline{\zeta}$ is equal to 163, and in Fig. 2, the integral over Γ_{coh} with respect to $\overline{\zeta}$ is equal to 0.45. It follows from Eq. (6.10) that in this case, the coherent energy loss will be greater than the incoherent loss when $N_{eg}>360$. Therefore coherent SASE is seen to be important for a short electron bunch with sharp ends. Such a case was studied experimentally at the Sunshine facility [17].

Our analysis of the incoherent SASE is similar to that presented in Ref. [13]. The coherent SASE was not considered in Ref. [13], but has been recently considered by Piovella [18], utilizing the frequency-domain equations of Ref. [11], which are related to the time-domain equations (3.4)–(3.6) by Fourier transform.

VII. RADIATED FIELD FROM BUNCH MUCH SHORTER THAN RADIATION WAVELENGTH

In Appendix B the linearized FEL equations are solved retaining the E^* term. The E^* term may be neglected for a long electron bunch, but must be retained when the electron bunch length is comparable to or shorter than the radiation wavelength. As a specific example, in Appendix B we have determined the radiated field from an electron bunch whose length is very short compared to the radiation wavelength. Here, we shall discuss this case in more detail.

As in Eq. (4.18), we take

$$-\infty\equiv\zeta_0(0)<\zeta_1(0)<\zeta_2(0)<\cdots<\zeta_N(0)<\zeta_{N+1}(0)\equiv\infty. \tag{7.1}$$

Since the bunch is very short compared to the radiation wavelength, to good approximation all $\zeta_j(0)\cong\psi$ and

$$e^{-i\zeta_j(0)}\cong e^{-i\psi}\quad (j=1,\ldots,N). \tag{7.2}$$

Inside the electron bunch, it is shown in Eq. (B21) that for $\zeta_j(0)<\zeta<\zeta_{j+1}(0)\leqslant\zeta_N(0)$

$$E(\tau,\zeta)\cong\frac{d_1k_0}{\gamma_0}\left[j+j^2\left(\frac{i\alpha k_0}{n_1}\right)\frac{1}{2}(\tau-\zeta+\psi)^2\right]e^{-i\psi}. \tag{7.3}$$

In this case $\zeta\cong\psi$. In front of the bunch, it is shown in Eq. (B22) that for $\zeta_N(0)<\zeta<\zeta_N(0)+\tau$

$$E(\tau,\zeta)\cong\frac{d_1k_0}{\gamma_0}\left[N+N^2\left(\frac{i\alpha k_0}{n_1}\right)\frac{1}{2}(\tau-\zeta+\psi)^2\right]e^{-i\psi}. \tag{7.4}$$

The field vanishes for $\zeta>\zeta_N(0)+\tau$. At the position of the jth charge, $E(\tau,\zeta_j(0))$ is determined by Eq. (A5) as the average of the field just in front of and just behind it. One finds

$$E(\tau,\zeta_j(0))=\frac{d_1k_0}{2\gamma_0}\left\{(2j-1)+[(j-1)^2+j^2]\times\left(\frac{i\alpha k_0}{n_1}\right)\frac{1}{2}\tau^2\right\}e^{-i\psi}. \tag{7.5}$$

The ponderomotive phase shift of the jth source is found by inserting Eq. (7.5) into Eq. (4.4), yielding $(j=1,\ldots,N)$

$$\delta\zeta_j(\tau)=-(2j-1)\left(\frac{\alpha k_0}{n_1}\right)\frac{1}{2}\tau^2. \tag{7.6}$$

It is straightforward to verify that the field as given in Eqs. (7.3)–(7.5) satisfies the partial differential equation

$$\left(\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\zeta}\right)E(\tau,\zeta)=\frac{d_1k_0}{\gamma_0}\sum_{j=1}^{N}e^{-i\psi}[1-i\delta\zeta_j(\tau)]\times\delta(\zeta-\zeta_j(0)). \tag{7.7}$$

In order to determine the correction to the radiated energy from coherent spontaneous emission, it is necessary to consider terms neglected in the linear approximation utilized in this paper. This problem will be considered in future work [19].

VIII. CONCLUSIONS

For a given initial set of longitudinally discrete radiators, we have solved the linearized one-dimensional FEL equa-

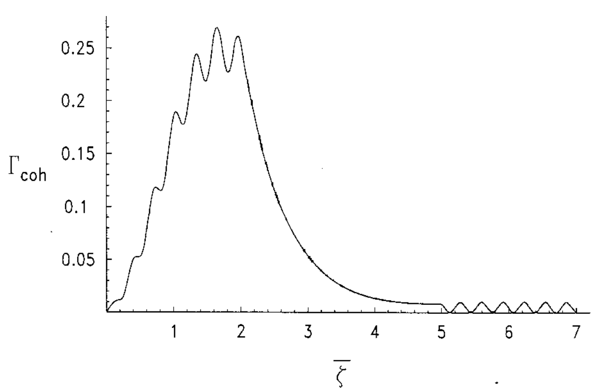

FIG. 2. We plot $\Gamma_{coh}(\overline{\tau},\overline{\zeta};\overline{\zeta_B},\rho)$ against $\overline{\zeta}$, for $\overline{\tau}=5$, $\overline{\zeta_B}=2$, and $\rho=\frac{1}{40}$. The integral under the curve is 0.45. The scaled quantities plotted are dimensionless.

tions to determine the evolution of the radiation field. As in previous studies of SASE [2–5,11], the discreteness of the radiators enters the initial conditions. A distinguishing feature of the present work is that the discreteness of the radiators has also been retained in the determination of the response functions $g_{1,2}(\tau,\zeta,\zeta')$. This has allowed us to elucidate the common practice of approximating the electron gain medium by a continuum fluid. We have been able to obtain the results of the continuum fluid approximation by carrying out the appropriate limit on explicit expressions we have derived for the response functions. In this paper we have treated the electrons as charge sheets, rather than as point particles. It is to be hoped that in the future, a three-dimensional approach will be developed, taking into account the full discreteness of point electrons. Such work would extend the three-dimensional descriptions of SASE presented in Refs. [4,5].

In treating the collective instability in FELs, it is usual to neglect the E^* term in the evolution equation (4.6). In Appendix B we have obtained the solution of Eq. (4.6) retaining the E^* term. Neglecting the E^* term yields the correct leading behavior when the electron bunch is long compared to the radiation wavelength and the Pierce parameter ρ is small. In this case, the response functions [Eqs. (4.29) and (4.30)] depend on the initial phases of the radiators only through the function $m(\zeta,\zeta')$, defined to be equal to the number of radiators between ζ' and ζ. On the other hand, when the E^* term is retained (Appendix B); nonleading terms in the response functions [Eqs. (B14) and (B15)] are found which exhibit a more complicated dependence on the initial phases. When the electron bunch has a length comparable to or shorter than the radiation wavelength, the E^* term is important. This will also be true for a longer bunch as saturation is approached, but of course in this case nonlinear effects must be taken into account.

ACKNOWLEDGMENTS

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APPENDIX A: PROOF OF EQ. (4.10)

We integrate the paraxial wave equation (3.6) over a small interval about $\zeta=\zeta_j(\tau)$:

$$\int_{\zeta_j(\tau)-\varepsilon}^{\zeta_j(\tau)+\varepsilon} d\zeta\left(\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\zeta}\right)E(\tau,\zeta) = \int_{\zeta_j(\tau)-\varepsilon}^{\zeta_j(\tau)+\varepsilon} d\zeta\; d_1k_0\sum_{l=1}^{N}\frac{e^{-i\zeta_l(\tau)}}{\gamma_l(\tau)}\,\delta(\zeta-\zeta_l(\tau)). \tag{A1}$$

Taking the limit $\varepsilon\to 0$, we find

$$E(\tau,\zeta_j(\tau)+)-E(\tau,\zeta_j(\tau)-)=\frac{d_1k_0}{\gamma_j(\tau)}\,e^{-i\zeta_j(\tau)}. \tag{A2}$$

Now we multiply Eq. (3.6) by $E(\tau,\zeta)$, and then integrate over the same interval about $\zeta=\zeta_j(\tau)$:

$$\int_{\zeta_j(\tau)-\varepsilon}^{\zeta_j(\tau)+\varepsilon} d\zeta\; E(\tau,\zeta)\left(\frac{\partial}{\partial\tau}+\frac{\partial}{\partial\zeta}\right)E(\tau,\zeta) = \int_{\zeta_j(\tau)-\varepsilon}^{\zeta_j(\tau)+\varepsilon} d\zeta\; d_1k_0\sum_{l=1}^{N}\frac{e^{-i\zeta_l(\tau)}}{\gamma_l(\tau)}\,E(\tau,\zeta)\,\delta(\zeta-\zeta_l(\tau)). \tag{A3}$$

Taking the limit $\varepsilon\to 0$, we obtain

$$\frac{1}{2}\,[E^2(\tau,\zeta_j(\tau)+)-E^2(\tau,\zeta_j(\tau)-)] = \frac{d_1k_0}{\gamma_j(\tau)}\,e^{-i\zeta_j(\tau)}E(\tau,\zeta_j(\tau)). \tag{A4}$$

Using Eq. (A2) in Eq. (A4) yields the desired result:

$$E(\tau,\zeta_j(\tau))=\tfrac{1}{2}[E(\tau,\zeta_j(\tau)+)+E(\tau,\zeta_j(\tau)-)]. \tag{A5}$$

In the case of rf accelerators, a relation of the type exhibited in Eq. (A5) has been called the ''fundamental theorem of beam loading'' by Wilson [20].

APPENDIX B: DISCUSSION OF THE E^* TERM

In discussing the dominant coherent growth in the FEL, we have neglected the term proportional to E^* on the right-hand side of Eq. (4.6). Let us briefly discuss the solution when the E^* term is not dropped. In this case, Eq. (4.19) is replaced by

$$f(s,\zeta_j(0)+)-f(s,\zeta_j(0)-) = \frac{2\sigma^2}{s^2}\,[f(s,\zeta_j(0))+f^*(s,\zeta_j(0))e^{-2i\zeta_j(0)}] + \frac{d_1k_0}{\gamma_0 s}\,e^{s\zeta_j(0)}e^{-i\zeta_j(0)}. \tag{B1}$$

Equation (4.20) is unchanged:

$$f(s,\zeta_j(0))=\tfrac{1}{2}[f(s,\zeta_j(0)+)+f(s,\zeta_j(0)-)]. \tag{B2}$$

Using Eq. (B1), its complex conjugate, and Eq. (B2), we derive

$$f(s,\zeta_j(0)+)=B_j(s)f(s,\zeta_j(0)-) + \frac{d_1k_0}{\gamma_0 s}\,e^{s\zeta_j(0)}e^{-i\zeta_j(0)}\Lambda(s), \tag{B3}$$

and Eq. (4.22) is still valid,

$$f(s,\zeta_j(0)-)=f(s,\zeta_{j-1}(0)+)+\int_{\zeta_{j-1}(0)}^{\zeta_j(0)} d\zeta' e^{s\zeta'}E_0(\zeta'). \tag{B4}$$

In Eq. (B3) we have defined

$$B_j(s)=\Lambda(s)+\Gamma(s)e^{-2i\zeta_j(0)}C, \tag{B5}$$

$$\Lambda(s)=1+\frac{2\sigma^2}{s^2}, \tag{B6}$$

$$\Gamma(s)=\frac{2\sigma^2}{s^2}, \tag{B7}$$

and the operator C, which takes the complex conjugate of all factors to its right. For example,

$$u_1Cu_2u_3=u_1u_2^*u_3^*C, \tag{B8}$$

$$Cu_1Cu_2u_3=u_1^*u_2u_3. \tag{B9}$$

As in Eq. (4.18), $-\infty\equiv\zeta_0(0)<\zeta_1(0)<\zeta_2(0)<\cdots<\zeta_N(0)<\zeta_{N+1}(0)\equiv\infty$.

With the initial condition, Eq. (4.23),

$$f(s,\zeta_0(0)+)=0, \tag{B10}$$

the recursion relations given in Eqs. (B3) and (B4) determine $f(s,\zeta_j(0)+)$ for all j. Then, $f(s,\zeta)$ can be found for arbitrary ζ by noting that for $\zeta_j(0)<\zeta<\zeta_{j+1}(0)$

$$f(s,\zeta)=f(s,\zeta_j(0)+)+\int_{\zeta_j(0)}^{\zeta}d\zeta' e^{s\zeta'}E_0(\zeta'). \tag{B11}$$

From Eqs. (B3), (B4), (B10), and (B11), we find for $\zeta_j(0)<\zeta<\zeta_{j+1}(0)$

$$\begin{aligned}f(s,\zeta)=&\sum_{l=1}^{j}B_j(s)B_{j-1}(s)\cdots B_l(s)\int_{\zeta_{l-1}(0)}^{\zeta_l(0)}d\zeta' e^{s\zeta'}E_0(\zeta')\\&+\int_{\zeta_j(0)}^{\zeta}d\zeta' e^{s\zeta'}E_0(\zeta')\\&+\frac{d_1k_0}{\gamma_0 s}\sum_{l=1}^{j-1}B_j(s)B_{j-1}(s)\cdots B_{l+1}(s)\\&\times e^{s\zeta_l(0)}e^{-i\zeta_l(0)}\Lambda(s)+\frac{d_1k_0}{\gamma_0 s}e^{s\zeta_j(0)}e^{-i\zeta_j(0)}\Lambda(s).\end{aligned} \tag{B12}$$

Note that since the operators $B_i(s)$ do not commute, the order of the factors in Eq. (B12) is important. The operators corresponding to larger phases appear to the left.

Equation (B12) can be simplified, by introducing the operator $\beta(s,\zeta,\zeta')$ defined by

$$\beta(s,\zeta,\zeta')=B_j(s)B_{j-1}(s)\cdots B_l(s), \tag{B13}$$

where the product is over all factors $B_i(s)$ corresponding to phases $\zeta_i(0)$ lying between ζ' and ζ, again with factors corresponding to larger phases appearing to the left. When there are no sources between ζ' and ζ, then $\beta(s,\zeta,\zeta')$ is unity. Following steps analogous to those leading from Eq. (4.25) to Eq. (4.30), we find

$$\begin{aligned}g_1(\tau,\zeta,\zeta_j(0))=&\theta(\zeta-\zeta_j(0))\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi is}\\&\times e^{s(\tau-\zeta+\zeta_j(0))}\beta(s,\zeta,\zeta_j(0)+)\Lambda(s)\end{aligned} \tag{B14}$$

and

$$g_2(\tau,\zeta,\zeta')=\theta(\zeta-\zeta')\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}e^{s(\tau-\zeta+\zeta')}\beta(s,\zeta,\zeta'). \tag{B15}$$

Recall that

$$B_j(s)=1+\frac{i\alpha k_0}{n_1s^2}+\frac{i\alpha k_0}{n_1s^2}e^{-2i\zeta_j(0)}C \tag{B16a}$$

and

$$\Lambda(s)=1+\frac{i\alpha k_0}{n_1s^2}. \tag{B16b}$$

From Eqs. (B13) and (B16), we see that in the continuum limit, $\Lambda(s)\to 1$ and

$$\beta(s,\zeta,\zeta')\to\exp_+\left\{\frac{i\alpha}{s^2}\int_{\zeta'}^{\zeta}d\zeta_0D(\zeta_0)[1+e^{-2i\zeta_0}C(\zeta_0)]\right\}. \tag{B17}$$

The ζ-ordered exponential $\exp_+(\ )$ employed in Eq. (B17) is defined such that factors depending on larger values of ζ appear to the left. The operator $C(\zeta_0)$ takes the complex conjugate of all factors appearing to its right.

Much work remains to explore the implications of the solution of the linearized FEL equations retaining the E^* term, which we have presented in this Appendix. This will be addressed in future [19]. The effect of the E^* term can be expected to be important when the bunch length is comparable to or shorter than the radiation wavelength. Let us consider the very special case of a bunch whose length is very short compared to the radiation wavelength. We suppose the length to be so short that in Eq. (B16a) we can use the approximation

$$e^{-2i\zeta_j(0)}\cong e^{-2i\psi}\quad(j=1,\ldots,N). \tag{B18}$$

We wish to determine $f(s,\zeta)$ from Eq. (B12), in the absence of an external field.

Using Eq. (B18), it follows that for $l=1,\ldots,j-1$

$$\begin{aligned}&B_j(s)B_{j-1}(s)\cdots B_{l+1}(s)\Lambda(s)e^{-i\psi}\\&\quad\cong\left\{1+[2(j-l)-1]\frac{i\alpha k_0}{n_1s^2}\right\}e^{-i\psi}.\end{aligned} \tag{B19}$$

Now, it follows from Eq. (B12) that for $\zeta_j(0)<\zeta<\zeta_{j+1}(0)$

$$f(s,\zeta)\cong\frac{d_1k_0}{\gamma_0 s}\left[j+j^2\left(\frac{i\alpha k_0}{n_1s^2}\right)\right]e^{-i\psi}. \tag{B20}$$

The inverse Laplace transform is taken using Eqs. (4.27) and (4.16). Inside the bunch, for $\zeta_j(0)<\zeta<\zeta_{j+1}(0)\leqslant\zeta_N(0)$

$$E(\tau,\zeta)\cong\frac{d_1k_0}{\gamma_0}\left[j+j^2\left(\frac{i\alpha k_0}{n_1}\right)\frac{1}{2}(\tau-\zeta+\psi)^2\right]e^{-i\psi}. \tag{B21}$$

Ahead of the bunch, for $\zeta_N(0)<\zeta<\zeta_N(0)+\tau$

$$E(\tau,\zeta)\cong\frac{d_1k_0}{\gamma_0}\left[N+N^2\left(\frac{i\alpha k_0}{n_1}\right)\frac{1}{2}(\tau-\zeta+\psi)^2\right]e^{-i\psi}. \tag{B22}$$

For $\zeta>\zeta_N(0)+\tau$, the field vanishes. At the position of the jth charge, $E(\tau,\zeta_j(0))$ is determined by Eq. (A5) as the average of the field just in front of and just behind it.

Now let us briefly discuss the case of a bunch long compared to the radiation wavelength. The fundamental resonance in the FEL corresponds to the electromagnetic (EM) wave (frequency ω_0, wave number k_0) slipping ahead of the electrons traveling at velocity $v_0=\omega_0/(k_0+k_w)$ by one radiation wavelength $2\pi/k_0$, as the electrons traverse one wiggler period. The E^* term provides a resonant interaction between the electron beam and a slow EM wave (frequency ω_0, wave number k_0+2k_w) of opposite helicity to the fast wave, which slips behind the electrons by one radiation wavelength $2\pi/(k_0+2k_w)$ as the electrons traverse one wiggler period. The slow wave is not present initially, and is not radiated spontaneously. It is, however, eventually generated by the interaction between the electron beam and the radiation. Once generated it can be amplified.

APPENDIX C: EVALUATION OF RESPONSE FUNCTION $g_1(\tau,\zeta,\zeta')$

Let us consider the representation for $g_1(\tau,\zeta,\zeta')$ given in Eq. (4.30):

$$g_1(\tau,\zeta,\zeta')=\theta(\zeta-\zeta')\times\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}\,e^{sx}\,\frac{s}{s^2-\sigma^2}\left(\frac{s^2+\sigma^2}{s^2-\sigma^2}\right)^{m(\zeta,\zeta'+)}, \tag{C1}$$

where we introduce the notation

$$x=\tau-\zeta+\zeta'. \tag{C2}$$

For a long wiggler, i.e., $2\rho\tau\gg1$, it is useful to follow Ref. [10] and obtain an asymptotic approximation for g_1 utilizing the saddle point method. We suppose τ and $\zeta-\zeta'$ are large, as is $m(\zeta,\zeta'+)$. We rewrite Eq. (C1) in the form

$$g_1(\tau,\zeta,\zeta')=\theta(\zeta-\zeta')\int_{c-i\infty}^{c+i\infty}\frac{ds}{2\pi i}\,\frac{s}{s^2-\sigma^2}\,e^{p(s)}, \tag{C3}$$

where

$$p(s)=sx+m(\zeta,\zeta'+)\ln\left(\frac{s^2+\sigma^2}{s^2-\sigma^2}\right). \tag{C4}$$

The equation determining the saddle point is $p'(s)=0$, which can be written as

$$s^4-\sigma^4=s_0^3s, \tag{C5}$$

where

$$s_0^3=\frac{4m(\zeta,\zeta'+)\sigma^2}{x}. \tag{C6}$$

Treating σ^2 as a small parameter, we solve Eq. (C5) iteratively to find the location $s=\hat{s}$ of the saddle point, obtaining

$$\hat{s}=s_0+\frac{\sigma^4}{3s_0^3}+\cdots. \tag{C7}$$

Substituting Eq. (C10) into Eq. (C7) yields

$$p(\hat{s})\approx\frac{3}{2}(4m\sigma^2)^{1/3}x^{2/3}\left[1+\frac{\sigma^4}{9}\left(\frac{x}{4m\sigma^2}\right)^{4/3}\right]. \tag{C8}$$

In Eq. (C8) we choose the cube root of σ^2 to have a positive real part, corresponding to coherent growth of the radiation field. Defining

$$y=\frac{k_0}{n_1}m(\zeta,\zeta'+), \tag{C9}$$

and using Eq. (4.13) in Eq. (C8), we obtain

$$p(\hat{s})\approx3\rho e^{i\pi/6}(2x^2y)^{1/3}\left[1+\frac{e^{i\pi/3}}{18(2^{1/3})}\left(\frac{\rho k_0}{n_1}\right)^2\left(\frac{x}{y}\right)^{4/3}\right] \tag{C10}$$

and

$$g_1(\tau,\zeta,\zeta')\propto e^{p(\hat{s})}. \tag{C11}$$

The first term in Eq. (C10) is the result obtained in the continuum limit [10]. The second term is a correction resulting from the discreteness of the radiators. We note that, as in Eq. (6.12),

$$N_{eg}=\frac{n_1}{2\rho k_0} \tag{C12}$$

is the (peak) number of electrons a radiation wave front slips over in the time it takes an electron to travel one gain length down the wiggler. The effect of discreteness will be negligible when

$$N_{eg}\gg1. \tag{C13}$$

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